# The Expected Variation of Random Bounded Integer Sequences of Finite Length[1]

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**Abstract**

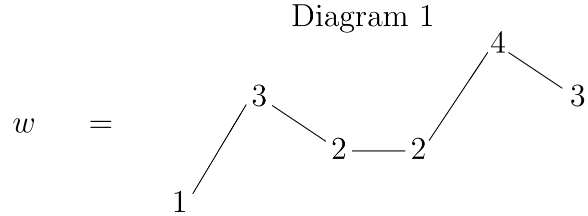
From the enumerative generating function of an abstract adjacency statistic, we deduce the mean and variance of the variation on random permutations, rearrangements, compositions, and bounded integer sequences of finite length.

**Key words** sequence variation, sequence oscillation, adjacency

## 1 Introduction

When the finite sequence of integers $w = 1, 3, 2, 2, 4, 3$ is sketched as below,

Diagram 1




its most compelling aspect is its vertical *variation*, that is, the sum of the vertical distances between its adjacent terms. Denoted by $\operatorname{var} w$, the vertical variation of the sequence in Diagram 1 is $\operatorname{var} w = 2 + 1 + 0 + 2 + 1 = 6$. Our purpose here is to compute the mean and variance of var on four classical sets of combinatorial sequences.

To formalize matters and place our problem in the context of other work, let $[m]^n$ denote the set of sequences $w = x_1x_2\dots x_n$ of length $n$ with each $x_i \in \{1, 2, \dots, m\}$. For a real-valued function $f$ on $[m]^2$, the *$f$-adjacency number* of $w = x_1x_2\dots x_n \in [m]^n$ is defined to be

$$\operatorname{adf} w = \sum_{k=1}^{n-1} f(x_k x_{k+1}).$$

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Some specializations of the $f$-adjacency number have been considered elsewhere. For instance, if $f(xy)$ is 1 when $x < y$ and 0 otherwise, then $\operatorname{adf} w$ is known as the *rise number* of $w$ [1, 3, 4]. For the selection $f(xy) = |y - x|$, $\operatorname{adf} w = \operatorname{var} w$. In a sorting problem of computer science, Levcopoulos and Petersson [5] introduced the related notion of oscillation ($\operatorname{var} w - n + 1$) as a measure of the presortedness of a sequence of $n$ *distinct* numbers. In [6], compositions were enumerated by their *ascent variation*, the $f$-adjacency statistic induced by $f(xy) = y - x$ if $x < y$ and 0 otherwise. For the case $f(xy) = h(|y - x|)$ where $h$ is a linear, convex, or concave increasing real-valued function, Chao and Liang [2] described the arrangements of $n$ distinct integers for which adf achieves its extreme values.

Besides considering the distribution of var on the set $[m]^n$, we also consider it on the set of rearrangements $R_n(i_1, i_2, \ldots, i_m)$ consisting of sequences of length $n = i_1 + i_2 + \cdots + i_m$ which contain $l$ exactly $i_l$ times, on the set of permutations $S_n = R_n(1, 1, \ldots, 1)$ of $\{1, 2, \ldots, n\}$, and on the set of compositions of $m$ into $n$ parts $C_n(m) = \{x_1 x_2 \ldots x_n \in [m]^n : x_1 + x_2 + \ldots + x_n = m\}$. For $m, n \geq 2$, Table 1 displays the mean and variance of var on these four sets. The $k^{th}$ falling factorial of $n$ is $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$, $\vec{i} = (i_1, i_2, \ldots, i_m)$,

**Table 1**

| Sequences | Expected Value of var | Variance of var |
|---|---|---|
| $S_n$ | $\dfrac{n^2-1}{3}$ | $\dfrac{(n-2)(n+1)(4n-7)}{90}$ |
| $[m]^n$ | $\dfrac{(n-1)(m^2-1)}{3m}$ | $\dfrac{(m^2-1)(6m^2n+6n-7m^2-2)}{90m^2}$ |
| $R_n(\vec{i})$ | $\dfrac{2}{n}\sum_{1\leq x<y\leq m}(y-x)i_x i_y$ | See Equation (12) |
| $C_n(m)$ | $\dfrac{2(n-1)}{(m-1)^{\underline{n-1}}}\sum_{x=1}^{\lfloor m/2\rfloor}(m-2x)^{\underline{n-1}}$ | See Equation (16) |

and, for $r$ a real number, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to $r$. The results in Table 1 are new. David and Barton [3, ch. 10] present the distributions of several statistics (some $f$-adjacency numbers, some not) primarily on permutations. We also note that Tiefenbruck [7] derived a generating function for compositions with bounded parts by a close relative of var. We leave open questions concerning the asymptotic behavior of var.

## 2 Enumerative factorial moments for $f$-adjacencies

Before working specifically with var, we discuss the enumerative generating function for adf on sequences as developed by Fedou and Rawlings [4]. Let $[m]^*$ denote the set of sequences of $1, 2, \ldots, m$ of finite length (including the empty sequence of length 0). For $w = x_1 x_2 \ldots x_n \in [m]^*$, we define $Z^w = z_{x_1} z_{x_2} \cdots z_{x_n}$. The enumerative generating function for adf over $[m]^*$ is then defined to be $G(p) = \sum_{w \in [m]^*} p^{\operatorname{adf} w} Z^w$.

By manipulating $G(p)$, we will obtain all of the information in Table 1 (and more). As a brief outline of our approach, note that the coefficient of $p^k z_1^{i_1} z_2^{i_2} \ldots z_m^{i_m}$ in $G(p)$ is just the number of rearrangements $w$ in $R_n(\vec{i})$ with $\operatorname{adf} w = k$. Thus, by dividing the coefficient of $z_1^{i_1} z_2^{i_2} \ldots z_m^{i_m}$ in $G'(1)$ by the cardinality of $R_n(\vec{i})$, we will obtain the mean of adf. So, in general, we compute the $d$th *enumerative factorial moment* $G^{(d)}(1) = \sum_{w \in [m]^*} (\operatorname{adf} w)^{\underline{d}} Z^w$.

From the work of Fedou and Rawlings [4], it follows that

$$G(p) = \frac{1}{D(p)} \tag{1}$$

where

$$D(p) = 1 - \sum_{n \geq 1} \sum_{x_1 \ldots x_n \in [m]^n} Z^{x_1 \ldots x_n} \prod_{k=1}^{n-1} (p^{f(x_k x_{k+1})} - 1). \tag{2}$$

Examples are presented in [4, 6] for which D has a closed form. We do not know of a closed form for D when adf = var (that is, when $f(x, y) = |y - x|$). Nevertheless, (1) is still useful in computing the mean and variance of var.

Although the formula for taking the $d$-fold derivative with respect to $p$ of a function of the form in (1) is known, we provide a short derivation. To avoid the quotient and chain rule, rewrite (1) as $GD = 1$. Differentiating the latter $d$ times, $d \geq 1$, and dividing by $d!$ gives

$$\sum_{j=0}^{d} \frac{G^{(d-j)}}{(d-j)!} \frac{D^{(j)}}{j!} = 0. \tag{3}$$

To solve for $G^{(d)}$, consider the system

$$\begin{array}{ccccccccccc}
\frac{G^{(d)}}{d!}\frac{D^{(0)}}{0!} & + & \frac{G^{(d-1)}}{(d-1)!}\frac{D^{(1)}}{1!} & + & \frac{G^{(d-2)}}{(d-2)!}\frac{D^{(2)}}{2!} & + & & \cdots + & \frac{G^{(0)}}{0!}\frac{D^{(d)}}{d!} & = & 0 \\
 & + & \frac{G^{(d-1)}}{(d-1)!}\frac{D^{(0)}}{0!} & + & \frac{G^{(d-2)}}{(d-2)!}\frac{D^{(1)}}{1!} & + & & \cdots + & \frac{G^{(0)}}{0!}\frac{D^{(d-1)}}{(d-1)!} & = & 0 \\
 & & & & & & & & \vdots & & \\
 & & & & & & \frac{G^{(1)}}{1!}\frac{D^{(0)}}{0!} & + & \frac{G^{(0)}}{0!}\frac{D^{(1)}}{1!} & = & 0 \\
 & & & & & & & & \frac{G^{(0)}}{0!}\frac{D^{(0)}}{0!} & = & 1
\end{array}$$

where the top $d$ equations arise from repeated application of (3). Cramer's rule applied to the above system yields

$$\frac{G^{(d)}}{d!} = \frac{(-1)^d}{D^{d+1}} \begin{vmatrix}
\frac{D^{(1)}}{1!} & \frac{D^{(2)}}{2!} & \frac{D^{(3)}}{3!} & \cdots & & \frac{D^{(d)}}{d!} \\
\frac{D^{(0)}}{0!} & \frac{D^{(1)}}{1!} & \frac{D^{(2)}}{2!} & & & \frac{D^{(d-1)}}{(d-1)!} \\
0 & \frac{D^{(0)}}{0!} & \frac{D^{(1)}}{1!} & & & \frac{D^{(d-2)}}{(d-2)!} \\
\vdots & & & & & \vdots \\
0 & & \cdots & 0 & \frac{D^{(0)}}{0!} & \frac{D^{(1)}}{1!}
\end{vmatrix}$$

which, when expanded, implies

$$G^{(d)} = \sum_{\nu=1}^{d} \frac{(-1)^{\nu}}{D^{\nu+1}} \sum_{\substack{j_1 + \cdots + j_\nu = d \\ j_k \geq 1}} \binom{d}{j_1 \ldots j_\nu} D^{(j_1)} \cdots D^{(j_\nu)}. \tag{4}$$

To determine the enumerative factorial moment $G^{(d)}(1)$, we see from (2) that

$$D^{(j)}(1) = -\sum_{r=2}^{j+1} D_r^{(j)} \tag{5}$$

where

$$D_r^{(j)} = \sum_{x_1 \dots x_r \in [m]^r} Z^{x_1 \dots x_r} \sum_{\substack{l_1 + \cdots + l_{r-1} = j \\ l_k \geq 1}} \binom{j}{l_1 \dots l_{r-1}} \prod_{k=1}^{r-1} f(x_k x_{k+1})^{\underline{l_k}} .$$

For instance,

$$\begin{aligned} D_2' &= \sum_{xy \in [m]^2} f(xy) z_x z_y, \quad D_2'' = \sum_{xy \in [m]^2} f(xy)^{\underline{2}} z_x z_y, \quad \text{and} \\ D_3'' &= 2 \sum_{vxy \in [m]^3} f(vx) f(xy) z_v z_x z_y. \end{aligned} \tag{6}$$

Further setting $\vec{j} = (j_1, \dots, j_\nu)$, $s(\vec{j}) = j_1 + \cdots + j_\nu$,

$$\binom{d}{\vec{j}} = \binom{d}{j_1 \dots j_\nu}, \quad \text{and} \quad D_\mu^{(\vec{j})} = \sum_{\substack{r_1 + \cdots + r_\nu = \mu \\ r_k \geq 2}} D_{r_1}^{(j_1)} \cdots D_{r_\nu}^{(j_\nu)},$$

it follows from (4) and (5) that

$$G^{(d)}(1) = \sum_{\nu=1}^{d} \frac{1}{D^{\nu+1}(1)} \sum_{\substack{s(\vec{j}) = d \\ j_k \geq 1}} \binom{d}{\vec{j}} \sum_{\mu=2\nu}^{d+\nu} D_\mu^{(\vec{j})}. \tag{7}$$

As $D(1) = 1 - (z_1 + \cdots + z_m)$, extracting the contributions made by all $w \in [m]^n$ from both sides of (7) gives the $d$th enumerative factorial moment of adf over $[m]^n$ as

$$\sum_{w \in [m]^n} (\operatorname{adf} w)^{\underline{d}} Z^w = \sum_{\nu=1}^{d} \sum_{\substack{s(\vec{j}) = d \\ j_k \geq 1}} \binom{d}{\vec{j}} \sum_{\mu=2\nu}^{d+\nu} \binom{n+\nu-\mu}{\nu} \left( \sum_{i=1}^{m} z_i \right)^{n-\mu} D_\mu^{(\vec{j})} \tag{8}$$

valid for $d \geq 1$. When $d = 1, 2$, (6) and (8) imply that

$$\sum_{w \in [m]^n} \operatorname{adf} w \, Z^w = (n-1) \left( \sum_{i=1}^{m} z_i \right)^{n-2} \sum_{xy \in [m]^2} f(xy) z_x z_y \tag{9}$$

and that

$$\sum_{w\in[m]^n} (\operatorname{adf} w)^{\underline{2}}\, Z^w = (n-1)\left(\sum_{i=1}^m z_i\right)^{n-2} \sum_{xy\in[m]^2} (f(xy))^{\underline{2}} z_x z_y$$
$$+2(n-2)\left(\sum_{i=1}^m z_i\right)^{n-3} \sum_{vxy\in[m]^3} f(vx)f(xy) z_v z_x z_y \qquad (10)$$
$$+(n-2)(n-3)\left(\sum_{i=1}^m z_i\right)^{n-4} \left(\sum_{xy\in[m]^2} f(xy) z_x z_y\right)^2 .$$

# 3 Discussion of Table 1

The entries in Table 1 are consequences of (9) and (10) with $f(xy) = |y-x|$ and with appropriate substitutions for $Z$. For the mean and variance of var on the set of bounded sequences $[m]^n$, put $z_i = 1$ for $1 \le i \le m$. Noting that

$$\sum_{xy\in[m]^2} |y-x| = \sum_{1\le x<y\le m} 2(y-x) = 2\binom{m+1}{3},$$

it follows from (9) that the mean of var on $[m]^n$ is

$$\frac{1}{m^n} \sum_{w\in[m]^n} \operatorname{var} w = \frac{2(n-1)m^{n-2}}{m^n}\binom{m+1}{3} = \frac{(n-1)(m^2-1)}{3m}.$$

As

$$\sum_{xy\in[m]^2} |y-x|^{\underline{2}} = 4\binom{m+1}{4}$$

and as

$$\sum_{vxy\in[m]^n} |x-v||y-x| = \sum_{1\le v<x<y\le m} 2(x-v)(y-x)$$

$$+ \sum_{1\le x<y\le v\le m} 4(v-x)(y-x) - \sum_{1\le x<y\le m} 2(y-x)^2$$

$$= \frac{7m^2-8}{10}\binom{m+1}{3},$$

(10) implies that

$$\frac{1}{m^n}\sum_{w\in[m]^n}(\operatorname{var} w)^{\underline{2}} = \frac{4(n-1)}{m^2}\binom{m+1}{4} + \frac{(n-2)(7m^2-8)}{5m^3}\binom{m+1}{3} + \frac{4(n-2)(n-3)}{m^4}\binom{m+1}{3}^2.$$

Then, subbing the last result into

$$\frac{1}{m^n}\sum_{w\in[m]^n}(\operatorname{var} w)^{\underline{2}} + \frac{(n-1)(m^2-1)}{3m} - \left(\frac{(n-1)(m^2-1)}{3m}\right)^2$$

and simplifying gives the variance of var as recorded in Table 1.

For $R_n(\vec{i})$, extracting the coefficient of $z_1^{i_1}z_2^{i_2}\cdots z_m^{i_m}$ from (9) leads to

$$\sum_{w\in R_n(\vec{i})}\operatorname{var} w = 2(n-1)\sum_{1\le x<y\le m}(y-x)\binom{n-2}{i_1\ldots i_x-1\ldots i_y-1\ldots i_m}.$$

As the cardinality of $R_n(\vec{i})$ is

$$\binom{n}{i_1\, i_2 \ldots i_m} = \binom{n}{\vec{i}},$$

it follows that the mean of var on $R_n(\vec{i})$ is

$$\binom{n}{\vec{i}}^{-1}\sum_{w\in R_n(\vec{i})}\operatorname{var} w = \frac{2}{n}\sum_{1\le x<y\le m}(y-x)i_x i_y. \tag{11}$$

Let $\vec{i}_{\backslash r} = (i_1, \ldots, i_r - 1, \ldots, i_n)$. For example, $(3,2,1,4)_{\backslash 3\backslash 2\backslash 3} = (3,1,-1,4)$. The variance on $R_n(\vec{i})$ is then

$$\binom{n}{\vec{i}}^{-1}\sum_{w\in R_n(\vec{i})}\operatorname{var} w^{\underline{2}} + \frac{2}{n}\sum_{1\le x<y\le m}(y-x)i_x i_y - \left(\frac{2}{n}\sum_{1\le x<y\le m}(y-x)i_x i_y\right)^2 \tag{12}$$

where, upon extraction of the coefficient of $z_1^{i_1} z_2^{i_2} \cdots z_m^{i_m}$ from (10), we have

$$\begin{aligned}\sum_{w \in R_n(\vec{i})} (\operatorname{var} w)^{\underline{2}} = (n-1) \sum_{1 \le x,y \le m} |y-x|^{\underline{2}} \binom{n-2}{\vec{i}_{\backslash x \backslash y}} \\ +2(n-2) \sum_{1 \le v,x,y \le m} |x-v||y-x| \binom{n-3}{\vec{i}_{\backslash v \backslash x \backslash y}} \\ +(n-2)(n-3) \sum_{1 \le u,v,x,y \le m} |v-u||y-x| \binom{n-4}{\vec{i}_{\backslash u \backslash v \backslash x \backslash y}}.\end{aligned} \tag{13}$$

The permutation entries in Table 1 follow from (11) and (12). Selecting $m = n$ and $i_k = 1$ for $1 \le k \le n$ in (11) reveals the mean of var on $S_n$ as

$$\frac{1}{n!} \sum_{w \in S_n} \operatorname{var} w = \frac{2}{n} \sum_{1 \le x < y \le n} (y-x) = \frac{2}{n} \binom{n+1}{3} = \frac{n^2-1}{3}.$$

From (13) with $m = n$ and $i_k = 1$ for $1 \le k \le n$,

$$\begin{aligned}\sum_{w \in S_n} (\operatorname{var} w)^{\underline{2}} &= (n-1)! \sum_{1 \le x,y \le n} |y-x|^{\underline{2}} \\ &\quad +2(n-2)! \sum_{1 \le v,x,y \le n} |x-v||y-x| \\ &\quad +(n-2)! \sum_{\substack{1 \le u,v,x,y \le m \\ \{u,v\} \cap \{x,y\} = \emptyset}} |v-u||y-x| \\ &= (4/15)(n-2)!(10n^2+14n-27) \binom{n+1}{4}.\end{aligned}$$

So the variance of var on $S_n$ is

$$\frac{1}{n!} \sum_{w \in S_n} \operatorname{var} w^{\underline{2}} + \frac{n^2-1}{3} - \left( \frac{n^2-1}{3} \right)^2 = \frac{(n-2)(n+1)(4n-7)}{90}.$$

For $w = x_1 \ldots x_n \in [m]^n$, let $||w|| = x_1 + \cdots + x_n$. For the composition results in Table 1, set $z_k = q^k$ for $1 \le k \le m$. Then (9) implies that

$$\sum_{w \in [m]^n} \operatorname{var} w \, q^{||w||} = (n-1) q^{n-2} \left( \frac{1-q^m}{1-q} \right)^{n-2} \sum_{1 \le x,y \le m} |y-x| \, q^{x+y} \tag{14}$$

and (10) leads to

$$\sum_{w\in[m]^n} (\operatorname{var} w)^{\underline{2}}\, q^{||w||} = (n-1)q^{n-2}\left(\frac{1-q^m}{1-q}\right)^{n-2} \sum_{1\le x,y\le m} |y-x|^{\underline{2}}\, q^{x+y}$$
$$+2(n-2)q^{n-3}\left(\frac{1-q^m}{1-q}\right)^{n-3} \sum_{1\le v,x,y\le m} |x-v||y-x|\, q^{v+x+y} \qquad (15)$$
$$+(n-2)(n-3)q^{n-4}\left(\frac{1-q^m}{1-q}\right)^{n-4} \sum_{1\le u,v,x,y\le m} |v-u||y-x|\, q^{u+v+x+y}.$$

Extracting the coefficient of $q^m$ from (14) to obtain

$$\begin{aligned}\sum_{w\in C_n(m)} \operatorname{var} w &= 2(n-1)\sum_{1\le x<y\le m} (y-x)\binom{m-1-x-y}{n-3}\\ &= 2(n-1)\sum_{1\le x\le \lfloor m/2\rfloor} \binom{m-2x}{n-1}\end{aligned}$$

and then dividing by the cardinality $\binom{m-1}{n-1}$ of $C_n(m)$ gives the mean of var as stated in Table 1. The variance is

$$\binom{m-1}{n-1}^{-1} \sum_{w\in C_n(m)} \operatorname{var} w^{\underline{2}} + \frac{2(n-1)}{(m-1)^{\underline{n-1}}} \sum_{1\le x\le\lfloor m/2\rfloor} (m-2x)^{\underline{n-1}}$$
$$-\left(\frac{2(n-1)}{(m-1)^{\underline{n-1}}} \sum_{1\le x\le\lfloor m/2\rfloor} (m-2x)^{\underline{n-1}}\right)^2 \qquad (16)$$

where, pulling the coefficient of $q^m$ from (15), we have

$$\sum_{w\in C_n(m)} (\operatorname{var} w)^{\underline{2}} = (n-1)\sum_{1\le x,y\le m} |y-x|^{\underline{2}}\binom{m-1-x-y}{n-3}$$
$$+2(n-2)\sum_{1\le v,x,y\le m} |x-v||y-x|\binom{m-1-v-x-y}{n-4} \qquad (17)$$
$$+(n-2)(n-3)\sum_{1\le u,v,x,y\le m} |v-u||y-x|\binom{m-1-u-v-x-y}{n-5}.$$

The sums in (17) simplify somewhat. For instance,

$$\sum_{1\le x,y\le m} |y-x|^{\underline{2}} \binom{m-1-x-y}{n-3} = 4 \sum_{1\le x\le \lfloor m/2\rfloor} \binom{m-2x}{n}.$$

As a part of the second sum on the righthand side of (17), we note that

$$\sum_{1\le v<x<y\le m} (x-v)(y-x)\binom{m-1-v-x-y}{n-4}$$
$$= \sum_{2\le x\le \lfloor (m+1)/2\rfloor} \left( \binom{m-3x+1}{n} - \binom{m-2x+1}{n} + x\binom{m-2x}{n-1} \right).$$

The four-fold sums arising in the last sum in (17) reduce to double sums.

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